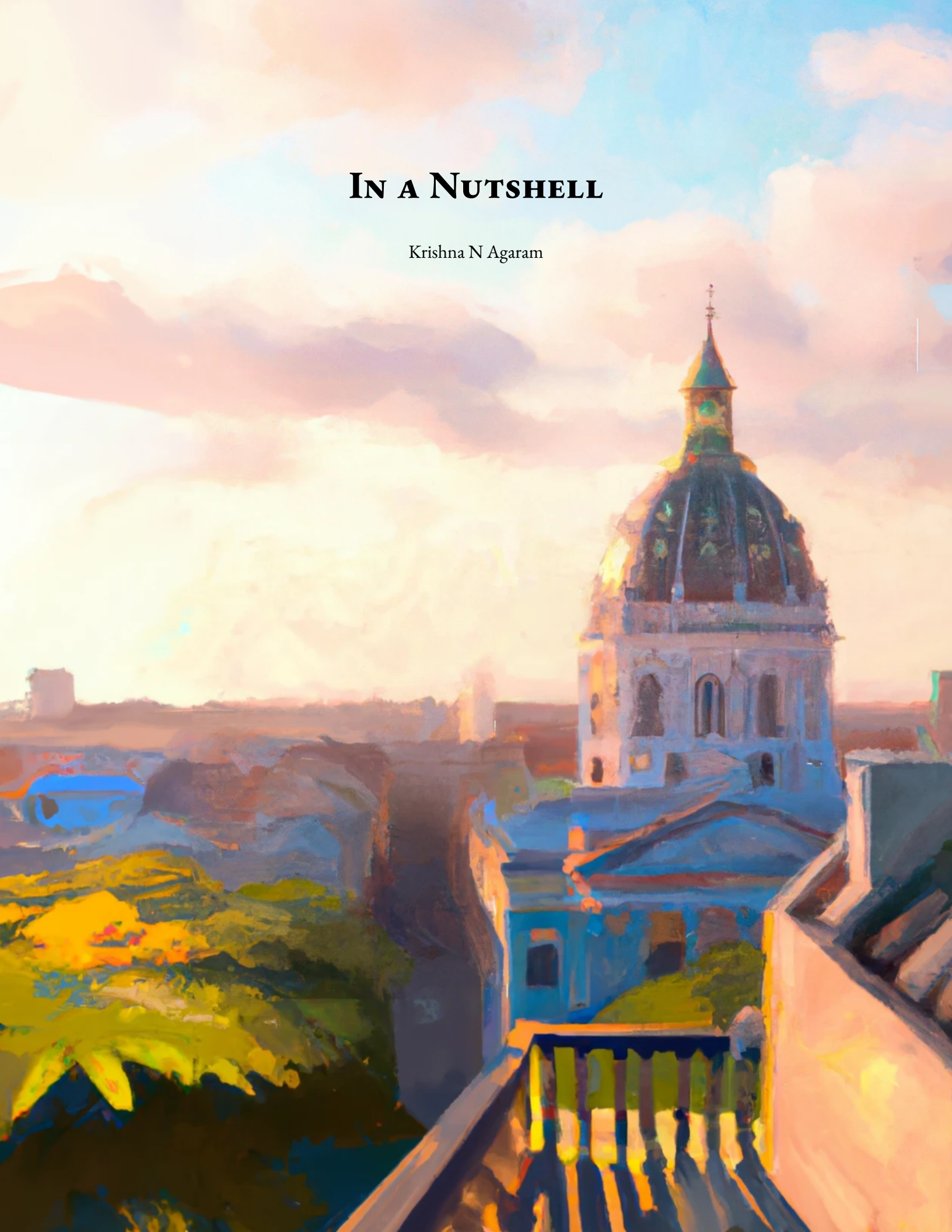
In a Nutshell

Krishna N Agaram



To

.

Author's Note: .

First edition, published in 2024.

.

Contents

Prologue

'So it's decided, then,' surmised Countess Aureola. The other, busy after a long day at the office, nodded. 'Go on.'

'We start out this evening, and tomorrow we meet P. The next day, we seek out the Indistinguishable, who should lead us to Comb Fort.' She stopped for breath. 'The twelve realms are then to be crossed. I wonder how long that will take, since we only have two weeks...'.

'We'll make it in good time,' the other assured the Countess. 'Hmm. After that, there's just the clothesline into the Enumerators' temple. That's all.'

'Deal. Let's get moving.'

Chapter 1

The Letter

'There are things in this world that people work on just for a certain childlike pleasure, you see,' said Dr. Mukund Rao as he and his son walked past Erdös waterfall on the way to the Ghoshs'. <describe environment>.

Dr. Ghosh had called up to say that he had proved the *Roy-Rényi* conjecture , and Dr. Rao, most interested to examine the proof that had long eluded them, was presently hurrying over. 'A true Enumerator gets his happiness simply when he counts correctly,' he continued.

Ganesh Rao, or simply Rao, as his friends called him, simply smiled . Unlike his and Rajan Ghosh's fathers, who were well-known Enumerators who spent hours carefully counting things, he hadn't the patience for that! Games were more his thing. And solving mazes. You couldn't really blame him for that, could you? He was only just going to be fifteen.

The party of two were nearing Rajan's house. 'Come, come,' echoed Dr. Ghosh's voice as he saw them from the porch. 'What worked?' Dr. Rao asked, unable to control his excitement.

'Singularity analysis of the associated OGF worked.'

The two Enumerators immediately went into the study, while Rajan and Ganesh were left to themselves in his room. 'Counting must be really fun, don't you think?' Rajan said,

once they started a game of chess. 'Well, I can't say for sure, I don't really know any.' 'Fair.' and there was silence for the rest of the game.

After a delicious snack of *Til Laddus* and *Murukku*, Dr. Ghosh called everyone to the living room.

'Ganesh, it's your birthday in a few weeks, isn't it?' Dr. Ghosh asked. Ganesh nodded. 'I have here a bag of chocolates, but I'm not sure it will be enough for all your friends. Let's see how many there are.'

'Oh, Thank you, Uncle. But please don't open the bag now, I'd like to open it on my birthday.'

'How will you count the chocolates, then? There is no number listed on the bag.'

'Oh!' Ganesh started, and then stopped. He wouldn't, of course, want to run out of chocolates. But he didn't want to open the bag now. He had to find another way to know how many there were. 'Don't worry, Ganesh, there's a way we can find out just how many there are.' Dr. Rao said, as though reading his thoughts. 'But how?' he asked.

'It tells you that there is one chocolate for each possible flavour you could think of.'

'But how many flavours are there?'

'It says that each flavour has a dark or milk chocolate shell, a vanilla or caramel filling, a banana or melon fruit punch,

Ingredient	Option1	Option2
Shell	Dark	Milk
Filling	Vanilla	Caramel
⋮	⋮	⋮

Figure 1.1: Choose any one of each ingredient, and you'll find a chocolate!

and, oh, two alternatives for each of four more chocolate ingredients. Do you see how you can use this knowledge?'

'We need to count just how many flavours this can make...'

'Exactly.'

Rajan, who was silent so far, spoke. 'Let's just count the number of milk chocolates, you know, for starters. Then for each flavor of milk chocolate, we can swap the milk for dark and get an equal number of dark chocolates! So we just need to double the number of milk chocolate flavors.'

'Very good,' Dr. Rao agreed. 'But we now have the same problem again, with but one less choice to make!'

Ganesh thought to himself. If one could look only at milk chocolates, one could also look only at caramel-filled milk chocolates. That should be even easier to count, since there were less of these. Plus, again, swapping the caramel for vanilla, the number of milk chocolates would just be twice the number of caramel-filled milk chocolates. And then?

Counting melon-caramel-milk chocolates? This swapping trick was nice. He could just multiply by two and then forget

about that ingredient, accounting for both choices for the ingredient. Could he keep multiplying by two? Hmm. 'I've got it!' he exclaimed. Everyone turned to him.

'Let's fix a choice for each of the seven things in the chocolate. Suppose that, er, could you pass me the bag, Appa?' – Dr. Rao obliged – 'Yes, so suppose we instead find the number of almond-topped blueberry-drizzled crunchy hazelnut-centred melon-punch caramel milk chocolates. That's just one, of course. And then there's one more chocolate with a walnut topping instead of almond, so there's two blueberry-drizzled crunchy hazelnut-centre melon-punch caramel milk chocolates.' he took a moment to collect his breath.

'Now, I can swap the blueberry drizzle for raspberry in each of these two, and we shall have four chocolates – almond blueberry, walnut blueberry, almond raspberry, walnut raspberry. There.'

'Yes, good, good, go on,' Dr. Ghosh said. 'And then since these are all crunchy, I'd create four more of these and then turn them from crunchy to creamy. That's eight hazelnut-centred melon-punch caramel milk chocolates possible.'

'Ah, I get it. So there are then sixteen melon caramel milks, and thirty-two caramel milks, and, ... a hundred-twenty-eight chocolates in all!' exclaimed Rajan.

'Exactly!'

Dr. Rao smiled. 'That was very quick, Ganesh. Good work.' Ganesh smiled. This was fun. Plus, he had more than enough for his friends. More than enough to give each of his friends two each. Would he give them each a milk and dark version of the same chocolate? Maybe change only the punch from melon to banana? Well, there was time to decide that!

<insert picture of the choice tree for this setting, with chocolate drawings on each node>

Dinner was quite chaotic, as is generally expected when a mathematician discovers a proof. The Enumerators discussed the possible consequences and generalizations of the proof over *Roti* and *Dal*. The boys gulped down substantially more than their fair share of delicious hot *Kheer* while discussing the latest Marvel movies – apparently, *Dr. Strange in the Quarry of Quasars* was a must-watch.

The doorbell rang just as it was turning half-past-eight. Dr. Ghosh went to the door. There was nobody there. 'Oh, what's this?' he exclaimed, picking up a small envelope that was tied to the stem of a plant. 'To Dr. Ghosh and Dr. Rao, from M.A.R.Y.A.M,' he read out.

The other three crowded around the table as Dr. Ghosh opened the envelope. 'Interesting. Have you heard of this organization?' Dr. Rao queried. 'Nope,' Dr. Ghosh replied. 'It seemes to be an agency from Planet Analytique,' Dr. Rao

said as he read through their Wikipedia page. 'Amazing. A message from Planet Analytique is always welcome!' Dr. Ghosh laughed.

Planet Analytique was heaven on earth for Mathematicians - it was home to the Abacus, Compass and Maple Valleys, after all. Legend had it that the people who worked there literally turned coffee into theorems!

The message ran as follows:

Dear Dr. Arindam Ghosh and Dr. Mukund Rao,

We hope this finds you well. There is very little time to explain; the gist is as follows:

A scroll has been found with the details of the legendary <artifact name> that can unlock the full potential of an Enumerator. The scroll says that it is hidden within the ancient Lovász Library on your planet. Our intel tells us that there is a party heading there this very instant – and we cannot run the risk of passing so important an object as this into the wrong hands.

Thus, we would like you to get there ASAP to collect the <artifact name> first. We are sure that it would be of great use to the Enumerator community you have on Planet Earth. Be warned, though; it is not an easy journey to the Library.

I hope you make a wise choice.

Your ally,
The <an expansion for MARYAM that captures being like the FBI> (M.A.R.Y.A.M),
Wolfram Estate,
Planet Analytique.

There was silence for a few minutes, as the four of them registered what they had just read. The true potential of an Enumerator! Dr. Ghosh thought. The idea was simply incredible! How an object could unlock it, he was not sure, but he sure had to find out! 'I'm up!' he said.

'Me too!'

'No boys, this is not a small journey and I am not sure you boys will be able to keep up,' Dr. Rao started. 'Dr. Ghosh and I will go.'

'Oh, please, Appa, I really want to come too.'

'I've told you Ganesh, it could be dangerous. I've only been near the Library once, and it was a week's journey from here and I was only half way there!'

'Okay, so it's decided then. We'll start early tomorrow, I expect, Rao?' Dr. Rajan confirmed.

'Yes. Could Ganesh stay here for the time we're away, since my wife will be at Nursing School for the next two weeks.'

'Of course. I'm sure the boys will have great fun together.'

'Let's get some sleep now. I'm sure tomorrow will be a long day.'

Once the boys were back in Rajan's room and a game of *Scotland Yard* had finally come to an end, Ganesh asked, 'Rajan, of course we're going too, right? This seems straight out of a novel, and I really cannot bear to miss it.' He made an Indiana Jones pose (frame it better).

'Imagine finding something as great as <artifact>. And being the first in the world to get there. We'll be famous! Maybe M.A.R.Y.A.M's head will call us up to Planet Analytique! What fun!'

'Hey, calm down, Rao! I don't think we are going to Planet Analytique anytime soon, Sir! But yes, I was thinking the same thing. It sure would be fun to race to the <artifact name>, especially since it's so important. The idea of a treasure hunt just gets my *josh* going. Perhaps we can sneak out and follow my father?' Rajan replied.

'Yes, I think so. Most movies have that - you always start the journey without telling anyone, announce yourself in between, and they will have to let you continue with them. Works every time.'

Ganesh made a dash to the dining room, where two backpacks were already packed. Dr. Rao was very well-prepared. It was his routine to plan each day the previous night, and his Sticky-Note-for-the-day was always next to his seat on the

breakfast table.

Ganesh went up to the table, and sure enough, he found the Sticky-Note. It read:

05:30 AM: At breakfast.
06:00 AM: Leave for the H.M. Bus - leaves 06:13AM from Erdös waterfall.
08:30 AM: Gate A. Boat leaves at 09.25AM.
10:00 AM: Follow 3rd Dyck path on (10,10) till Lagos bridge. Rendezvous with P.
13:00 PM: Lunch.
14:00 PM: Leave to Comb. Need to be there by nightfall else dangerous.
19:30 PM: Meet contact at Comb. Dinner.
21:00 PM: Bed. Big day ahead. Realm 1/12.

After swiftly making a copy of the schedule for himself, Ganesh ran back upstairs. 'I've got the plans. Tomorrow, we start at 5:30 AM. Rajan, do you believe it, we are actually going on a quest! Oh, I'm sure this is going to be the most fun I've ever had!'

Rajan was equally excited. 'I can scarcely believe it myself.' After a haphazard fifteen minutes of packing, both boys crashed to their beds, dreaming of adventure. They dreamed of robots, airforts, jungles, and of course, mazes<some places they will actually be in>.

Little did they know they would actually be in some of these. Little did they know was in store for them. Oh, the adventure was just beginning! Rao and Rajan were going to be in the thick of things very, very soon.

Chapter 2

Into the Maze

Rajan almost turned off his alarm when it rang. They were going on an adventure, and in just one hour the bus would leave from Erdös waterfall! Oh, he had to get up now. 'We've to get going, Rao, up!'

Both boys scrambled to get ready and scarcely reached breakfast at 06.03 AM. The two Enumerators were, as planned, already on their way to the bus stop. Rajan's mother, Meena, was surprised to see Rajan and Rao. 'Well, boys, you are up quite early today?'

'Yes *Amma*, we are planning to follow *Appa* into the ... oh, nothing, we just need to meet P near Erdös waterfall ... I mean Prashant called us yesterday night. He wanted to show us something this morning,' Rajan stammered, as he realized what he'd done.

'Well, *kanna*, you don't need to hide it from me. I saw Ganesh yesterday night copying out his father's Sticky-Note.' Ganesh started to protest, but was cut short.

'I'm not saying no just yet, boys. I've thought about it, and Ganesh's Mother and I think it'll be an interesting adventure for you two after that long quarantine during the pandemic. But be forewarned: it is a *very* long trek to the Library! Oh, and you must promise to keep your phones with you at all times.'

Both boys heaved a sigh of relief. 'We promise! Thank you so much, *Amma*, we'll be quite careful. Besides, both of us are big boys now!'

'I've packed you guys some food already - it's in the boxes next to the fridge - and you had better hurry now. The next bus leaves at 06.20. Oh, and also, take some paper with you. I've heard that's incredibly useful in the Maze.' Meena finished with a slight smile. 'Take that last fact from experience. *Appa* and I had a hard time there without paper, the one time we'd been there.'

Rajan packed the food, while Rao put a book and some pencils in his bag. 'Thank you so much, *Aunty*, it is really kind of you to let us go on this adventure. We will be really careful!' Rao said. 'I really hope we get to that <artifact name> first'.

They were now at the door. 'Take care, boys, and give me or *Appa* a ring if anything goes wrong.' Both boys nodded assent, and they were off to adventure!

They barely caught the bus. Mostly because it was two minutes late. 'That was closer than I would've liked!' Rajan sighed. 'I agree. At least we made it. The Gods are with us.'

The nearly two-hour long ride to the outskirts of the city was uneventful, except for the signboards, which were quite amusing. Alon Avenue, Boole Boulevard, Clay County and Dyck District, indeed. And of course, the rest of the alphabet

followed. Ramanujan City ((Ou)R City, in short) was truly wonderful after the advent of the Enumerators.

It was 8.25AM when they finally alighted at Entrance A, David Hilbert Maze. They now had to locate one of their fathers, hoping they had not left for the lake yet. The entrance to the Maze - more precisely the entrance to the port at the lake that led to the Maze - was a huge place, much like any big airport.

'Ah, there they are!' Rao exclaimed. Through the large glass pane at the entrance, the boys saw the two Enumerators heading towards a signboard for Boarding Gate 37. The boys rushed in. And then rushed out the next minute. Apparently, Gate A was only on special permission, and the boys had nothing of the sort!

They hurried to Gate B and purchased their tickets. 'That has lightened my pocket considerably,' Rao grumbled, as they walked towards the gates. 'Oh, wait a second, where do we go?' Rajan exclaimed. Their boarding card just had the following cryptic message: "Gate: deacb"

'What now? We're getting late!" Rao complained. A quick chat with a kind official showed them the way to go - 'Ah, but yes, your gate is hidden amongst these seemingly useless characters. Let me explain.'

'Suppose you write down all the 5-letter words you could posibly make using one each of an a, b, c, d and e - they

needn't make any English sense - in the order they would appear in a dictionary (that is, alphabetically). Your gate is then simply the position where "DEACB" appears in this list. The staff here are quite clever - they do not need to pre-decide a gate for you, they just find you this word using some complicated function of your names, and you can then deduce the gate.'

'I see. Thanks a lot, Sir, we'll try to take it from here.' Rajan said. 'Good luck! Remember to be clever - you do not have all the time in the world to write down all the words before you miss your ship.'

'I wonder what he meant by don't write all of them down. I mean, isn't that the best way to go about it?' Rao glanced at his watch. 'Let's hurry.' The boys feverishly started writing.

Ganesh asked: 'Wait a second - how do we write five-letter words in dictionary order while being sure that we haven't missed any?'

'Hmm. Suppose we first fix the first three characters to be `abc` - every word starting with some other three characters comes after every word starting in `abc` in dictionary order, so it's fair to write these first - and then put in `d` and `e` in the other two positions left in the word to get `abcde` and `abced`.'

Ganesh agreed: 'Right. And after that we have exhausted words starting in `abc`. Well, time to move to `abd` - that's next

smallest. That leaves c and e for the other two positions, giving us the words abdce and abdec. And then we move to abe.'

'But what next? After abe?'

'By then, we would have actually counted all possible words with the first *two* letters being ab, right? So we'd now have to move to ac - that's next smallest after words beginning in ab. The first-three-letter possibilities we have to check in turn are now acb, acd and ace.'

'Ah, true. So after ab, we go through to ac, and then words beginning in ad and ae. And then we've exhausted every possible word starting in a! So we must next write words beginning with b - the first two letters look like ba, bc, bd and be, in this order. And for ba, we must work through words starting in bac, bad and bae. Then bc. And so on. Wow, our gate deacb is quite far behind in this dictionary!'

'Let's get to work. We haven't got much time.'

After a minute:

```
abcde
abced
abdce
 bdec
  ecd
  edc
```

```
acbde
acbed
 cdbe
  deb
```

To save some time, they stopped bothering about writing the first few letters because they stayed the same for so long! Why, oh why, were there so many words one could make with just five letters?

Another two minutes later:

'Oh, when will we ever get to `d`?' Rajan whined. 'This is quite pointless... Oh wait, hey, see this, there's a pattern. Every time that we just move the last three characters around and keep the first two the same, there are exactly 6 words we make. Like this: `cde ced dce dec ecd edc`.'

Rao, who had been writing the words starting with `b`, confirmed. 'Indeed, yes. Oh, I can explain it. You give the third position to any of the three letters left, say `c`, and the fourth to any of the remaining two, say `e`. This forces the word to be `ced`. We can change the `c` to `d` and to `e`. For each, we can choose any one of the two remaining letters to give us the fourth characters, and the word is forced. So there are $3 \times 2 = 6$ words with these three characters.'

'Oh, right. I think I get it, let me think for a minute.'

Third Letter	Fourth Letter	Word
c	d	cde
c	e	ced
d	c	dce
d	e	dec
e	c	ecd
e	d	edc

'Oh yes, I hadn't noticed it. We could, actually, do this for the last four letters too. Say the first letter is fixed as a. We have four more places to fill with four letters. Pick up one of the four letters for the second position in the word - we had put b there to start with - and for *each* of these four letters in the second place, there are $3 \times 2 = 6$ words that we could make with the remaining three letters.' Rajan was excited.

'And since this happens four times, one for each choice of the second letter, that gives us $4 \times 6 = 24$ words starting with first letter a in total.'

'You've got it!' Rao squeaked, much to the surprise of an old man who was heading past them. 'That's what the official meant. Count, but without listing all of them. We've got 24 words that start in a, and 24 starting in b or c. That's 72 so far. The first letter is now d. Okay, we've got to be careful. We cannot miss deacb. It's below $72 + 24 = 96$, at least, since it's somewhere among the words that start with d.'

Rajan nodded. 'Yeah. Okay, now let's write it this way. We

have to still run through da, db and dc. They're all before de, and each of them gives 6 words, as we noticed.'

```
a____ - 24
b____ - 24
c____ - 24
da___ -  6
db___ -  6
dc___ -  6
       ----
        90
```

'We're almost there. After these 90 we have deabc, and then, deacb! Gate 92, here we come!' Rao completed. Rajan was standing up. 'Wow, that was quite a neat thing too! If we'd known how to do this before, we would have got it by the time we got out of that stupidly long security queue! Anyways, we'd better hurry now!'

'My father always tells me - there is great adventure in doing something the first time; afterwards, it becomes but a *trivial* exercise - I guess it applies here too!' Rao laughed. Till today, he would've never thought one could get away with finding positions like this without writing out the whole list! This was plain magical.

The gate led them, across a small pier, to a medium-sized ferry. Hilbert Maze a thirty-minute ride away across Lake Lovász. The tall greenish-grey walls, littered with a variety

of mosses and climbers both fascinated and frightened the boys. This was some serious stuff! How they were to make their way through the maze they hadn't the faintest idea!

'I think the best idea is to regroup with Dad, once we get to HM. They will take us along with them from there, since we have already come this far,' Rajan mused, as they walked up the stairs into the boat. 'Yes,' Rao agreed.

About thirty boats set out at 9.25AM for Hilbert Maze. The Enumerators were on one of those, and the boys needed to locate them quickly - calls did not typically work in the middle of a lake. An uneventful half-hour saw them dash to the exits after the boat docked. They made their stand next to one of them, observing the crowd that came out.

'*Appa*, here!' Rajan waved his hands wildly, as he saw Dr. Ghosh and Dr. Rao come out of a nearby exit. 'Well, she certainly did well, since they're here now,' Dr. Rao was saying, and Dr. Ghosh chuckled.

'Well boys, why are you here?'

'Uh...we followed you here, *Appa*, not wanting to miss out on an adventure as big as this.'

Dr. Ghosh did not seem pleased. 'Well, I suppose we can't do much else except take you folks along. Or you could just head back home, I'd say that's the better option.' A cry of "No!" echoed from Rao and Rajan in unison.

'Okay, come along, but you take care of yourselves. You have your phones with you, right?' Both nodded assent. 'P is waiting for us. We haven't much time. Mukund, could you open up the Dyck path manual, please? We need the 3rd on (10, 10).'

The walls of the maze looked much taller when inside it. One appreciated how high seven stories tall really was. An hour of walking through the maze guided by Dr. Rao's phone brought them to a bridge across a fast rapid. 'Lagos, at last.' he said. He signalled to a small booth a little ahead of the end of the bridge. 'And finally, we shall meet P.'

Chapter 3

Gateway to the Twelve Realms I - meeting P

PERMute-IT, or PERMIT, as it was called, was a robot that operated the booth. It was, incidentally, also the enigmatic P - everyone knows PERMIT is too long to pronounce anyway. Everybody that wanted to cross over to the next section of the maze had to get past P.

Rao was a bit underwhelmed. 'This is P? Can't be!' he snorted. Dr. Rao grinned.

'Well, indeed it is, son. Now go along, and answer border control before we can move on. You do know that the region ahead is actually not part of RMS?'

'Oh, I see. *Appa*, could you please go first, I'd like to see how to do this.' Dr. Rao nodded. 'Sure. It's not all that hard, just be confident.'

Where were they headed, actually? They were exiting their home country, Ramanujan Metropolitan State (or RMS in short). They were headed to the country on the other side of P - quite unfancifully named Hilbert Maze, or HM. (The running joke among mathematicians was that RMS was always greater than HM. Haha!)

Dr. Rao went up to P, and filled out a form about why he was visiting HM. P asked him some questions, and then opened the gates. It was Rao's turn next. He went forward

nervously.

'Hello, there. Your name?'

'Ganesh Rao.'

'Reason for visit?'

'I want to visit the Lovász Library.'

'I see. Fill this form out. Good luck, and PERmute-IT!'

Ganesh took the form from P. It had three questions on it. The header said - "Please prove authenticity of visit. Answer the following to prove interest in moving on."

Okay, that couldn't be so bad. But wait a second! These were counting questions! He read the first question:

"How many ways can you put three people in three chairs arranged in a line?"

Oh, this was the same as the boarding gate riddle. One simply had to look at the first chair, put someone in there - there's 3 ways to do that - and then, for *each* such way, get one of the two remaining folks into the second chair - so that's 2 choices we have - and then the third chair automatically goes to the remaining person. A total of 6 ways to put them all in.

<the decision tree picture with pictures of the incomplete chair arrangement inside each node. Should make it very clear.>

He answered 6 next to the question. He move on to the

next question. "Can you solve the above problem with n people into three chairs, where n can now be any positive integer at least 3?"

Wait a minute. How could n people fit in three chairs? Ah, but they don't. Only three lucky people got to sit. He tried the same old idea again - n choices for our first chair. And for each of these choices, $(n-1)$ for the second chair. That's $n(n-1)$ ways to fix up the first two chairs. Was that right? He thought for a minute, then nodded. Yes, it looked alright.

For each of these $n(n-1)$ allotments, we have $(n-2)$ choices for who gets the third chair. That meant $n(n-1)(n-2)$ ways to set up all three chairs.

He was about to write it in, but he decided to check it for a value of n to make sure he was correct. Since he had already solved the case of $n = 3$, he tried that. $3 \cdot 2 \cdot 1 = 6$, it worked! He marked it in.

The last and final question read:

"Rewrite the above question's answer in terms of factorials. The **factorial** of number n is

$$\boxed{n! = n \times (n-1) \times (n-2) \times ... \times 1.}$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$."

Aha. So you just take number n, and keep subtracting by 1 as you multiply them all together.

Indeed, the previous answer $n(n-1)(n-2)$ looked tantalizingly similar to $n!$, but there was something off ... it didn't go all the way to 1. How could he fix that? He tried a few small numbers. For $n = 5$, it was $5 \cdot 4 \cdot 3$. It's just missing a $2 \cdot 1$.

He thought, well, let me give it that $2 \cdot 1$:

$$5 \cdot 4 \cdot 3 = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(2 \cdot 1) = 5!/(2 \cdot 1).$$

But hey, $2 \cdot 1$ is also a factorial - the factorial of 2. So $5 \cdot 4 \cdot 3$ was just $5!/2!$. That was sweet. Onto $n = 6$. $n(n-1)(n-2)$ was $6 \cdot 5 \cdot 4$. Again, Rao gave $3! = 3 \cdot 2 \cdot 1$ to complete $6!$. So $6 \cdot 5 \cdot 4$ was $6!/3!$. He tried the *general* case, where he left n as is to be any positive number. Well, what did one have to multiply to $n(n-1)(n-2)$ to get a factorial? $(n-3)(n-4)(n-5)...1$, of course. And that was exactly $(n-3)!$ itself. That meant -

$$\begin{aligned} n(n-1)(n-2) &= \frac{n(n-1)(n-2)(n-3)(n-4)...1}{(n-3)(n-4)...1} \\ &= \frac{n!}{(n-3)!} \end{aligned}$$

And there you go! The answer to the second question just in factorials was on the right-hand side! He was done. He handed in the filled form, P glanced at it. Satisfied, P replied: 'You are PERMITted to go. I hope you don't mind the pun, Sir!' Ganesh laughed. 'Thanks, P!'

He was quite happy with the way he attacked the questions. And how simple they worked out to be in the end!

The other important thing was, he'd gotten through, and the Library now awaited!

Before going through the gate, Rao quickly discussed the questions with Rajan, who was next. They might be useful, who knows? P handed him a similar form. The only difference though, was that 3 was now a variable itself, k. 'You get a slightly more *general* set, kid, since you now know the answer to a *particular* case - the case $k = 3$ - from your friend.'

Rajan sighed. It was probably a harder set, then. But he would try his best. What could possibly go wrong?

It was just as P promised. Where there was a three, there was a k.

The Questions.

1. How many ways can you put k people in k chairs arranged in a line?
2. Solve the same problem with n people into k chairs, where n can now be any positive integer at least k?
3. Rewrite the above question's answer in terms of factorials. I'm not going to define it again, since Rao must've told you already.

First question first.

The trick was to treat k exactly like you would 3, except that he would have to *think* a little more *generally*. Just as

before, there are k choices for the first chair, $(k-1)$ for the second for each choice of the person in the first chair, $(k-2)$ for the third for each of the $k(k-1)$ arrangements of people in the first two chairs.

The difference, now, was that there could be chairs left! The fourth chair could go in $(k-3)$ (the "-3" since three people are already sitting) ways for each arrangement of the first three chairs, so that's $k(k-1)(k-2)(k-3)$ ways to get the first four guys in. Well, I know what happens next! Rajan smiled.

We'll get to multiply $(k-4)$, $(k-5)$, $(k-6)$, and so on till you get to 3, 2, 1 when there's only 3, 2 and finally, 1 chair left to allot. The answer, then, was $k(k-1)(k-2)(k-3)(k-4)(k-5)...3 \cdot 2 \cdot 1$ which was – drumroll please – exactly $k!$. Neat.

The second was quite similar, only one had to be a little careful with when the last chair got occupied. Rajan did it the following way (transcribed as he spoke aloud): To start with, I'm going to have n choices for the first chair, $(n-1)$ for the second for each choice of the first, and so on.

How many choices for the last chair? - I need to know this because I've only got to multiply terms till there - there are no more chairs and choices after that. Hmm. When I get to the third chair, I used $(n-2)$ since that's the number of people left to be alloted (two are already sitting). When I get

to the kth (and last) chair, $(k-1)$ are already sitting, so there are $n-(k-1)=n-k+1$ choices for the last. And I must multiply until here, so that gives me the answer

$$n(n-1)(n-2)...(n-(k-2))(n-(k-1))$$

or

$$n(n-1)(n-2)...(n-k+2)(n-k+1).$$

Done!

The third question, since he knew the idea from Rao, was simple. What should you multiply the answer with to get $n!$? The answer has decreasing numbers from n to $n-k+1$.

So we need $n-k$, $n-k-1$, and all the way to 3, 2 and 1. That is, we need to multiply by $(n-k)!$. Thus, this was all:

$$\begin{aligned} & n(n-1)(n-2)...(n-k+2)(n-k+1) \\ =& \frac{n(n-1)(n-2)...(n-k+2)(n-k+1)(n-k)...2\cdot 1}{(n-k)!} \\ =& \frac{n!}{(n-k)!} \end{aligned}$$

And done!

As soon as he left P, Rajan said, 'Hey, Rao, look here! The answers I got to the general question were not much harder to find, but seem to be more powerful! Because now you can

put a 4, a 5, or whatever else you like for k and it'll always work!'

'Hmm, you're right! Variables do make life easier - solve it with the variable in there, get the answer in terms of the variables, and then later plug in whatever you want for it!'

The rest of the meeting with P was uneventful. Dr. Ghosh, of course, passed with flying colors in under one minute. The whole group was now officially in another country.

'Congratulations you two, you did remarkably.' Dr. Ghosh smiled. 'Indeed, good job, boys,' Dr. Rao agreed. 'Shall we picnic here for lunch?'

There was, of course, unanimous agreement.

Over lunch, the Enumerators explained what Ganesh and Rajan had managed to solve was. Mathematicians had already looked at this sort of this, it seemed. Apparently each arrangement of people into chairs was called a **permutation** of the people into places. And what the two boys had found was the *number* of different permutations of n people into k places - fancifully written ${}^{n}P_{r}$.

<define permutation in box and say we computed #perms = nPk. Also elaborate kPk = k! (question 1) and so 0! = 1>

Chapter 4

Gateway to the Twelve Realms II - Fort Comb

'Well, boys, we now come to one of the most scenic parts of the journey. Fort Comb is perhaps the most beautiful place I have ever seen.' Dr. Rao stared with a slight smile into the distance, no doubt reliving his first visit to the Fort.

Beyond P was the maze, again. Another mass of greenish-grey walls. Ganesh was excited. It was fun going off into the unkwown, especially when he was perfectly safe with his father by his side.

Dr. Ghosh's phone rang. It was Meena. 'Hello? Yes, we met them. We're past P heading to Comb...okay, got it. See you.'

An hour and a half passed as they walked through the seemingly never-ending maze. It was then that Rajan realized that something was off. 'Appa, are the walls supposed to get shorter as we go ahead? Because I'm quite sure this wall was much taller at the start of the maze.'

'Indeed, Rajan. That's the speciality of the Maze - the walls were built to look to be the same height throughout the Maze, even though the ground may be at a higher altitude.' Dr. Ghosh confirmed. 'We've been steadily walking uphill all this while, and it is going to get steeper soon. The Fort is at a great altitude, you see.'

<Fort Comb is really high, like a city in the clouds type feel>

Ganesh wondered whether he would be taller than the walls themselves by the time they reached Fort Comb. Little did he realize then that the danger in the climb to the fort was this very fact!

A round of chocolate cookies at 5.30pm rejuvenated the tired travellers. The top of the walls were nearly within arm's reach. Dr. Rao said: 'I think we should hurry now. The Indistinguishables will awaken soon.' Dr. Ghosh nodded with a worried expression. They did not want to rush into a group of Indistinguishables at this time - they were extremely unpredictable. Plus, the shrinking walls made them sitting targets.

It was nearing 7.00 pm when they hit a roadblock. The walls of the Maze in front of them had collapsed in, leaving a conical pile of rubble on the path in front of them. It was going to be difficult to get through that. 'Bother! This is an Indistinguishable's doing. We'd better hurry.' Dr. Ghosh started to climb up the six-foot tall pile.

The boys clambered up after him, with Dr. Rao taking the rear. It was slow progress, since the stones were loose - pulling the wrong ones could cause a whole lot of stones to come crashing down. They were almost at the top when Dr. Rao screamed. 'They are here. We need to run.' Dr.

Ghosh and the boys raced down the other side. Dr. Rao was just reaching the top when his foot slipped. He came down with a groan on the opposite side of the pile. 'Appa! Are you okay?' Ganesh screamed from the other side. 'I'm fine. The Indistinguishables are here though - we were too late. Put on a confident face and answer everything they ask carefully.' Dr. Rao replied.

A pack of around eight to ten Indistinguishables hovered above the pile. Each of them looked like a gigantic blue dragonfly. Each of them was around seven feet in length, with a wingspan of twelve feet. The uncanny thing was that each one looked *exactly* identical! - hence the name given to them. They examined their prey for a few minutes, saying nothing. One of them finally said - 'Why are you out at this time?'

Dr. Ghosh was prepared. 'We have permission to enter Fort Comb. The pile came in our way. If you will allow us, we need to get to the Fort quickly.' Another Indistinguishable spoke: 'You are Enumerators? We have orders to prosecute all Enumerators heading this way.'

Dr. Rao asked: 'Who do you work for?'

'We cannot tell. But you will come with us now.'

'No, we cannot. We have a mission to accomplish. And you will not stop us.' The Indistinguishables were not happy. They looked at each other. The next moment was unanticipated. Four of them swiftly picked up the humans before

they had any time to react and started flying towards Fort Comb. 'Hey, let us go!' Rajan tried to wriggle out of the iron grip of his captor. 'It's no use, son. Wait it out.' Dr. Ghosh shouted in the darkness.

Dr. Rao was thinking hard. He had read about a way to dispel Indistinguishables while at Planet Pseudorandom. What was it, now? He tried to remember. It was something about singing. Ah, right. They loved singing as a choir. The whole community seemed to share this interest (after all, they were Indistinguishables) and thus choir parties were most common.

His thoughts were interrupted by the creak of the seventy-foot tall gate of the Palace of Indistinguishability. The Palace stood fifty metres above ground in an equilibrium with the hot steam from the large geyser underneath. Clouds formed as the steam hit the cold buildings, giving the Palace the appearance of a city-in-the-clouds.

The gate slowly opened, and the four humans' mouths with it: the palace was breathtakingly beautiful. The palace comprised thirteen buildings, arranged as two concentric squares and three outposts on the three sides not facing the gate, and coloured by height in sky blue, silver and gold. The hallway leading into the palace was a sight to behold. Large golden chandeliers hung above the shiny spotless marble floor, and the walls were covered in tapestries depicting

the many victories of the Indistinguishables.

'Wow, this is beautiful! I have never seen something like this before,' Ganesh exclaimed. 'Wait a minute. This is just like Comb, right?' Dr. Ganesh asked Dr. Ghosh. 'Indeed, I think it's an Indistinguishable joke again, copying the architecture of Fort Comb,' came the reply.

'Cut it. Let's move,' said the Indistinguishable that was holding Dr. Ghosh and swiftly landed onto the hallway. The other Indistinguishables followed suit. The four were led down the length of the hallway to the throne room. The throne room was enormous. The height of three buses stacked on top of each other, it looked like a large auditorium. At the head of the room on a small throne sat an Indistinguishable (who need not be described, because well, he looked exactly the same as the others). 'King Sephon,' Ganesh's captor bowed. 'We have brought them.'

'Thank you, Poshen. I really hope that they will be able to do it.'

'Yes, my Lord.' The humans' captors left, leaving them alone with the king. Dr. Ghosh started: 'Your Highness, please let us be. We need to get to Comb fast.'

'Haha. Haha. Fort Comb, indeed,' Sephon started. 'Not till you get us what we want, Ghosh.' Dr. Ghosh was surprised. 'How do you know who I am?' Sephon smiled. 'It is my business to know things. Let us get down to work. Come

with me.'

The party followed King Sephon down a flight of stairs, and into what looked like a mini Hilbert Maze, except it was made of mirrors. 'They even copied the Mirrors,' Dr. Rao grumbled. After fifteen minutes of bumping against mirrors and navigating through the maze, they reached a large door. Sephon used his wingprint and opened it. 'This is our concert hall,' he explained. 'We will be hosting our first decennial choir contest here next week.'

'Why are you showing this to us?' Dr. Ghosh questioned. 'Because this is where we need your help.'

The four people looked at each other. 'What can we offer you, your Highness?'

'You are Enumerators, I take it? We need you to solve a problem for us that will help us plan the contest.'

'Oh. Sure, what is the problem?' Dr. Ganesh smiled for the first time since his fall from the mound of stones.

'It is simple. Each choir will have four people in a line, one mic to each singer.' Sephon began. Dr. Ghosh nodded. 'Go on.' Sephon continued: 'The goal of the contest is to find the best choir across our kind. Which means that every set of four Indistinguishables will come together for the contest, and we will pick the best choir that results.' He stopped to catch his breath. 'The only issue is that we don't know how

many choirs will result, and thus how long the contest will last. And that is where we need an Enumerator to help.'

Dr. Ghosh and Dr. Rao looked at each other and smiled. So this was the reason they were brought to the Palace. They conferred in hushed tones for a minute. Dr. Ganesh was heard saying, 'Yeah, they will get it,' with Dr. Ghosh nodding.

'So, will you help us?'

Dr. Ghosh replied: 'Our sons will help you, your Highness. They can solve it for you.'

'Thank you. You may explore the palace till they do so, and then we will let you all go. But note that a wrong answer will be punished. If we run out of time or have much time remaining at the contest next week, you shall not be spared.'

Ganesh and Rajan gulped. 'Appa, why don't you solve it for them? You guys are the Enumerators, after all,' Ganesh called after his father. 'Don't worry Ganesh, try your best. You will get it. We will be back in an hour, and confirm your answer when we are back. There is nothing to worry about, just enjoy trying it,' Dr. Rao reassured.

The boys heaved a sigh of relief. 'Come on boys, let's get going. I need to give you some more details first.'

'There are thirty-six of us here at the palace. Every four of us will sing for one minute at the contest. Note that it doesn't matter if Poshen gets the first mic, Nespoh the second,

Opshen the third and me the fourth, or any other order. We all look the same, and our combined voice will be too. The four of us should count as one performance. Now to the question: I want you to tell me how long the contest will last.'

The boys nodded. 'We understand, your Highness.' 'Good. I'll be back in an hour. Feel free to use any of the couches.' Sephon disappeared into the mirror maze.

'Well, what have we got into this time?' Ganesh laughed. Rajan laughed too. 'It seems like a people into chairs problem again. Thirty-six people vying for four mics - or chairs. I wonder when we will get done with people trying to sit in chairs.'

'Agreed. Let's write that down.'

$$\#\text{performances} = \frac{36!}{(36-4)!} = \frac{36!}{32!}$$

'Oh, hold on,' Ganesh continued. Does this not count these performances as different?

Left	Mic 1	Mic 2	Mic 3	Mic 4	Right
(1)	Poshen	Nespoh	Opshen	Sephon	
(2)	Sephon	Opshen	Nespoh	Poshen	

'Oh, yes! You're right. The first one is when Poshen gets Chair 1, and then Nespoh is chosen for the second, Opshen

for the third and Sephon for the fourth chair. And clearly, the second one where Sephon gets the first, Opshen the second, and then Nespoh and Poshen, is counted much later, and as different.'

'King Sephon said they were to all be counted as one performance. How do we remove this recounting?'

<box where this is called overcounting, where you count the same item once. Simialrly undercounting where you dont count a valid item.>

They thought about it for fifteen minutes, with not much luck. 'The problem is that we can't really choose them one after the other, because that'll count those two as different.'

'Oh, could we do this? We can write a performance as a thirty-six letter word of 0s and 1s, where there are only four 1s at the four positions corresponding to the singers,' Rajan though. 'So, the first performance above would be written like this, if we assume Poshen is the sixth citizen, Nespoh the nineteenth, Opshen the twenty-fifth and Sephon the twenty-sixth.'

000001000000000000100000110000000000

'And yes, the seond performance will give the same word, so whatever the order in which the four singers take the mics, their "word" will remain the same!'

'Oh, yes! So the number of performances is simply the

number of these 36-character words with exactly four ones,' Ganesh agreed.

'Only thing, we haven't any idea how to find that!' Rajan thought. And the two of them continued their attacks on the problem.

After a while, Rajan had another idea: 'Suppose we only had two singers per choir. That would be 36!/2! performances with re-counting. But every actual performance is counted *exactly* twice - the pair of singers A and B is counted as performances AB and BA (AB means that A gets the first mic, and B the second), which means our answer is precisely double the actual answer!'

$$\#\text{performances(2 per choir)} = \frac{1}{2} \cdot \frac{36!}{34!}$$

'Good gracious, you're right! *Every actual performance is counted the same number of times in our answer* - which means we can simply divide our answer by the number of times we could each actual performance to get the exact number of distinct performances!' Ganesh exclaimed. 'Surely this is not special to two singers per choir? What about three per choir?'

'Well, suppose that the three people in the performance are A, B and C. In our count, since we look at who gets which mic/chair, we are going to have 6 performances starring A, B and C, as these: ABC, ACB, BAC, BCA, CAB, CBA.

Which are -'

'Precisely all the ways in which one can arrange three letters A, B and C in three chairs - 3! = 6' Ganesh finished. 'Each performance is repeated exactly 3! times, because that's precisely how many times a performance can be recounted, and we count all of these. The answer is then:'

$$\#\text{performances(3 per choir)} = \frac{1}{3!} \cdot \frac{36!}{33!}$$

Rajan nodded. They were getting someplace. 'Now to four people per performance. A, B, C and D. Who picks which mic doesn't matter in actuality. Our count of 36!/32! counts each performance from the lens of who goes where mattering. It counts $ABCD$, $ABDC$, ... , $DCAB$, $DCBA$, all as different. There are 4! of these orderings of A, B, C and D within the mics. We need to divide our count by this, to count each performance not 4! times, but exactly once.'

They wrote down the final answer:

$$\#\text{performances(4 per choir)} = \frac{36!}{4!32!}$$

They high-fived each other. It did look like they had solved it, and they had done it with twenty minutes to spare. Ganesh noticed something here: 'The two numbers in the denominator actually add up to the one on the top: $4 + 32 = 36$.

And this happened in the two and three-per-choir cases too. Not a coincidence, I think?'

Rajan agreed. There was nothing special about 36 - he was quite sure 37 would also give this nice property. 'Well, for 37, it would be:'

$$\#\text{performances (37 total, 4 per choir)} = \frac{1}{4!}\cdot\frac{37!}{(37-4)!} = \frac{37!}{4!33!}$$

'...and it does work here too: $4 + 33 = 37$.' Ganesh was interested. This was neat. But he wanted to convince himself it would always be the case: 'Hey Rajan, since we have some time left, why don't we solve it with 36 replaced by n and 4 by k? It would solve any such problem simply by putting in values for n and k, plus we can check if the sum-of-denominator-equals-numerator thing holds for any n and k.'

Rajan agreed. 'Yes, let's'. Ganesh went on: 'Our count for n Indistinguishables into k chairs (since there are now k mics), would be ${}^{n}P_{k} = n!/(n-k)!$. Each performance of k people $A_1, A_2, A_3, ..., A_k$ would be counted as many times as they can be arranged into those k mics: which is $k!$. We divide, and this is what we get:'

$$\#\text{performances } (n, k) = \frac{n!}{k!(n-k)!}$$

'So it does hold in general: $k+(n-k)$ is always n.' Ganesh shouted in delight. He continued: 'We actually did the exact same thing as for 36 and 4, right? We just wrote n for 36 and k for 4, and got a more general statement.'

'Exactly. That's something I really liked across this journey. There's nothing special in a few numbers. The same trick would work for any pair of numbers.'

'Well, we just need to compute this number, and then we can go mess around in the mirror maze till our fathers and King Sephon get back.'

They got down to computing the number. In a few minutes, they had:

$$\begin{aligned}
\text{Answer} &= \frac{36!}{4!32!} \\
&= \frac{36 \times 35 \times 34 \times 33 \times \cancel{32!}^{1}}{4!\cancel{32!}^{1}} \\
&= \frac{\cancel{36}^{3} \times 35 \times \cancel{34}^{17} \times 33}{\cancel{4} \times \cancel{3} \times \cancel{2}} \\
&= 3 \times 35 \times 17 \times 11 \\
&= 19635
\end{aligned}$$

'That is a LOT of performances!' Ganesh exclaimed. So it will take, let's see -'

$$\begin{aligned}\#\text{Days} &= \frac{19635 \text{ minutes}}{1440 \text{ minutes/day}} \\ &= 13.6 \text{ days}\end{aligned}$$

'...two weeks to complete the whole thing, spending the whole night as well. Nineteen thousand is huge,' Rajan completed.

In a Nutshell

Join Rao and Rajan on the adventure of a lifetime!